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Electromagnetic first-order conservation laws in a chiral medium

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Abstract. The electromagnetic first-order conservation laws associated with the symmetric zilch Z and its antisymmetric companion Y previously derived for a field in a normal medium are extended to the case of a chiral medium.

1. *Introdution*

Recently Bailyn and the author [1] have established the electromagnetic first-order conservation laws associated with Lipkin's symmetric zilch Z [2] and its antisymmetric companion Y [3] for a field in a normal (homogeneous, isotropic and linear) medium with permitivities ϵ and μ. In this paper we extend the study to the case of a chiral medium. The conservation laws will be derived for a medium at rest.

2. The basic equations

We shall be concerned with chiral media characterized by the Fedorov constitutive relations [4]

$$D = \epsilon E + \epsilon\beta \nabla \times E \tag{1}$$

$$B = \mu H + \mu\beta \nabla \times H \tag{2}$$

in the rest frame of the medium. The pseudoscalar β measures the degree of chirality. With equations (1) and (2), Maxwell's equations in the absence of charges ($\nabla \cdot D = 0$, $\nabla \cdot B = 0$, $\nabla \times H = D_{,t}$, $\nabla \times E = -B_{,t}$, where $\partial a/\partial t = a_{,t} = \dot{a}$) give rise to the following equations for E and B:

$$\nabla \cdot E = 0 \tag{3}$$

$$\nabla \cdot B = 0 \tag{4}$$

$$\nabla \times E = -B_{,t} \tag{5}$$

and

$$\nabla \times B = \mu J + \epsilon\mu E_{,t} \tag{6}$$

where

$$J = \epsilon\beta \nabla \times K \tag{7a}$$

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with

$$K = 2E_{,t} + \beta\nabla \times E_{,t} = 2E_{,t} - \beta B_{,tt}. \tag{7b}$$

We then see that (3)–(6) are formally similar to Maxwell's equations for a normal medium with current densities $\rho = 0$ and J. If we take the curl of (5) and (6) we obtain the equations

$$\left(\epsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right) E = -\mu\frac{\partial J}{\partial t} \tag{8}$$

and

$$\left(\epsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right) B = \mu\nabla \times J. \tag{9}$$

3. The conservation laws

The fifteen conservation laws for the normal medium [1] involved the pseudovectors $E\times E_{,t}$ and $B \times B_{,t}$, and the pseudo-tensor $\dot{B}_i E_j - B_i\dot{E}_j$. The symmetric and antisymmetric combination of the first two quantities correspond to Lipkin's zilch Z^{oio} [2] and its partner Y^{oio} [3], respectively, and the i, j symmetric and antisymmetric combination of the last one correspond to Z^{ijo} and Y^{ijo}, respectively. Let us see what the expressions involving these quantities are in our case. If we calculate $(E \times E_{,t})_{,t}$ and use (8) we obtain the relation $(\partial a/\partial x_j = a_{,j})$

$$[E \times (\epsilon\mu E_{,t} + \mu J)]_{,t} = (E \times E_{,j})_{,j} + \mu(E_{,t} \times J). \tag{10}$$

We shall now show that the second term on the right-hand side of this equation is a divergence, as is the first one. Using (7*a, b*) we have

$$\frac{1}{\epsilon\beta}(E_{,t} \times J)_i = \dot{E}_j(K_{j,i} - K_{i,j}) = (\dot{E}^2)_{,i} - \beta[(\dot{E}_j.\dot{B}_{j,t})_{,i} - \dot{E}_{j,i}\dot{B}_{j,t}] - (\dot{E}_jK_i)_{,j} \tag{11}$$

where, in the last step, use has been made of (3). The second term inside the square bracket can be written as

$$\dot{E}_{j,i}\dot{B}_{j,t} = (\dot{E}_i\dot{B}_{j,t})_{,j} \tag{12}$$

which follows from (5), $(E_{j,i} = E_{i,j} - \epsilon_{ijk}\dot{B}_k)$ together with (4). This shows that $E_{,t} \times J$ is in fact a divergence and, therefore, that (10) can be written as a differential conservation law. To be consistent with the notation used in [1] we write it as

$$X^{ioo}{}_{,t} + X^{ioj}{}_{,j} = 0 \tag{13}$$

where, using (6),

$$X^{ioo} = (E \times (\nabla \times B))_i \tag{14}$$

and

$$X^{ioj} = -(E \times E_{,j})_i - \epsilon\mu\beta[(\dot{E}^2 - \beta\dot{E}_k\dot{B}_{k,t})\delta_{ij} + \beta\dot{E}_i\dot{B}_{j,t} - \dot{E}_jK_i]. \tag{15}$$

The pseudovector X^{ioo} is then to be interpreted as the density of the conserved quantity and X^{ioj} as expressing its flux. The constant of motion is then

$$X^{io} = \int X^{ioo}\mathrm{d}^3x. \tag{16}$$

Now we calculate $(B \times B_{,t})_{,t}$. Using (9) we get

$$\epsilon\mu(B \times B_{,t})_{i,t} = (B \times B_{,j})_{i,j} + \mu B_j(J_{j,i} - J_{i,j}). \tag{17}$$

The last term can be written as

$$B_j(J_{j,t} - J_{i,j}) = (\boldsymbol{B}\cdot\boldsymbol{J})_{,i} - (B_iJ_j)_{,j} - \epsilon\mu(\boldsymbol{J}\times\boldsymbol{E}_{,t})_i - (B_jJ_t)_{,j} \tag{18}$$

where we have used (4), (6) ($B_{j,i} = B_{i,j} + \epsilon_{ijk}(\mu J_k + \epsilon\mu\dot{E}_k)$), and $J_{j,j} = 0$ which follows from ($7a$). As we saw before, the term ($\boldsymbol{J}\times\boldsymbol{E}_{,t}$) can be written as a divergence. Therefore, all the right-hand side of (18) and consequently of (17) can be written as a divergence. We then have another conservation law. We get

$$X^{oio}{}_{,t} + X^{oij}{}_{,j} = 0 \tag{19}$$

where, with (5),

$$X^{oio} = -\epsilon\mu(\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{E}))_i \tag{20}$$

and

$$\begin{aligned} X^{oij} = &-(\boldsymbol{B}\times\boldsymbol{B}_{,j})_i - \mu\left\{(\boldsymbol{B}\cdot\boldsymbol{J} + \epsilon^2\mu\beta(\dot{E}^2 - \beta\dot{E}_k\dot{B}_{k,t}))\delta_{ij} - B_iJ_j\right. \\ &\left.-B_jJ_i + \epsilon^2\mu\beta(\beta\dot{E}_i\dot{B}_{j,t} - \dot{E}_jK_i)\right\}. \end{aligned} \tag{21}$$

The new conserved quantity is then

$$X^{oi} = -\epsilon\mu\int \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{E})\mathrm{d}^3x\,. \tag{22}$$

Finally we use (8) and (9) to write the relation

$$[\epsilon\mu\dot{B}_iE_j - B_i(\epsilon\mu\dot{E}_j + \mu J_j)]_{,t} = (E_jB_{i,k} - B_iE_{j,k})_{,k} + \mu[(\boldsymbol{\nabla}\times\boldsymbol{E})_iJ_j + (\boldsymbol{\nabla}\times\boldsymbol{J})_iE_j]\,. \tag{23}$$

Now we transform the last term. Using ($7a$) and noting from ($7b$) that $K_{j,j} = 0$, its first part can be written as

$$\frac{1}{\epsilon\beta}(\boldsymbol{\nabla}\times\boldsymbol{J})_iE_j = -(K_{i,k}E_j)_{,k} + 2\dot{E}_{i,k}E_{j,k} - \beta(\dot{B}_{i,t}E_{j,k})_{,k} + \beta\dot{B}_{i,t}\nabla^2E_j\,. \tag{24}$$

We also have from ($7a$, b), (5) and (3),

$$\frac{1}{\epsilon\beta}(\boldsymbol{\nabla}\times\boldsymbol{E})_iJ_j = \dot{B}_i(2\dot{B}_{j,t} + \beta\nabla^2\dot{E}_j)\,. \tag{25}$$

Before adding these two equations we use, for the second term on the right-hand side of (24), the relation

$$\dot{E}_{i,k}E_{j,k} = (\dot{E}_{k,i}E_j + \dot{E}_iE_{k,j})_{,k} - \dot{E}_{k,i}E_{k,j} + \delta_{ij}\dot{B}_k.\dot{B}_{k,t} - \dot{B}_i\dot{B}_{j,t}\,. \tag{26}$$

This follows from $E_{j,k} = E_{k,j} + (E_{j,k} - E_{k,j}) = E_{k,j} - \epsilon_{kjm}\dot{B}_m$ and similarly for $E_{i,k}$, and by making use of (3). Finally we write the second term on the right-hand side of (26) as

$$\dot{E}_{k,i}E_{k,j} = \tfrac{1}{2}[(\dot{E}_kE_{k,j})_{,i} + (\dot{E}_{k,i}E_k)_{,j} - (E_kE_{k,ij})_{,t}]\,. \tag{27}$$

Using this and the previous relation we can see that the sum of (24) and (25) can be written as a divergence plus a time derivative. In other words (23) can be written as a differential conservation law. We obtain

$$X'^{ijo}{}_{,t} + X'^{ijk}{}_{,k} = 0 \tag{28}$$

where, with (5) and (6),

$$X'^{ijo} = -\epsilon\mu(\boldsymbol{\nabla}\times\boldsymbol{E})_iE_j - B_i(\boldsymbol{\nabla}\times\boldsymbol{B})_j - \epsilon\mu\beta(E_mE_{m,ij} + \dot{B}^2\delta_{ij} + \beta\dot{B}_i\nabla^2E_j) \tag{29}$$

and

$$X'^{ijk} = B_i E_{j,k} - E_j B_{i,k} + \epsilon\mu\beta(K_{i,k}E_j + \beta\dot{B}_{i,t}E_{j,k} - 2\dot{E}_{k,i}E_j - 2\dot{E}_i E_{k,j} + \delta_{ik}\dot{E}_m E_{m,j} + \delta_{jk}E_m\dot{E}_{m,i}) . \quad (30)$$

The new set of (nine) constants of motion is then

$$X'^{ij} = \int X'^{ijo} \mathrm{d}^3x . \quad (31)$$

This, together with (16) and (22), give our fifteen constants of motion for the chiral medium. To conform with the notation in [1] we reserve the denomination $X^{ij\alpha}(\alpha = 0-4)$ for the quantity $X^{ij\alpha} = X'^{ij\alpha} - \delta_{ij}(\delta_{mn}X'^{mn\alpha})/2$, which, of course obeys the same differential conservation law expressed in (28). Instead of (31) we can take as constants of motion the quantities

$$X^{ij} = \int X^{ijo}\mathrm{d}^3x \quad (32)$$

where $X^{ijo} = X'^{ijo} - \delta_{ij}(\delta_{mn}X'^{mno})/2$, with X'^{ijo} given by (29).

The constants of motion X^{io} and X^{oi} in (16) and (22) have the same form as those derived for the normal medium but not so far X^{ij}. We mention that the symmetric and antisymmetric combination of the first two quantities give, respectively, the constants of motion associated with Lipkin's symmetric zilch [2] $Z^{io} = X^{io} + X^{oi}$, and its antisymmetric companion [3] $Y^{io} = X^{io} - X^{oi}$. The corresponding quantities constructed with X^{ij} are $Z^{ij} = X^{ij} + X^{ji}$ and $Y^{ij} = X^{ij} - X^{ji}$. Lipkin's Z^{oo} corresponds to $-X'^{mn}\delta_{mn}$, which is not an independent quantity.

4. Conclusions

We have extended the first-order conservation laws previously obtained for a field in a normal medium [1] to the case of a chiral medium. The first two sets of constants of motion, in (16) and (22), have exactly the same form as for the normal medium [1] but not the set in (32). The conservation laws have been established in the rest frame of the medium. A covariant approach, giving the results in an arbitrary frame, is presently under investigation.

In a normal medium [1] it is apparent from the covariant approach that we have only fifteen first-order conservation laws bilinear in the fields, since they are the fifteen independent components of a traceless 4-tensor $\overline{X}^{\alpha\beta}$, $\overline{X}^{\alpha}{}_{\alpha} = 0$. The laws considered here correspond one by one to those contained in $\overline{X}^{\alpha\beta}$, and reduce to them in a medium at rest when the chirality parameter β vanishes. Therefore, fifteen first-order conservation laws ('first-order' indicating that they involve first-order derivatives of the field quantities) is also all that we have in a chiral medium. This should be more apparent in the covariant approach to be considered in a forthcoming paper.

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